

Exact Cosmological Solutions of $f(R)$ Theories via Hojman Symmetry

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ABSTRACT

Nowadays, $f(R)$ theory has been one of the leading modified gravity theories to explain the current accelerated expansion of the universe, without invoking dark energy. It is of interest to find the exact cosmological solutions of $f(R)$ theories. Besides other methods, symmetry has been proved as a powerful tool to find exact solutions. On the other hand, symmetry might hint the deep physical structure of a theory, and hence considering symmetry is also well motivated. As is well known, Noether symmetry has been extensively used in physics. Recently, the so-called Hojman symmetry was also considered in the literature. Hojman symmetry directly deals with the equations of motion, rather than Lagrangian or Hamiltonian, unlike Noether symmetry. In this work, we consider Hojman symmetry in $f(R)$ theories in both the metric and Palatini formalisms, and find the corresponding exact cosmological solutions of $f(R)$ theories via Hojman symmetry. There exist some new solutions significantly different from the ones obtained by using Noether symmetry in $f(R)$ theories. To our knowledge, they also have not been found previously in the literature. This work confirms that Hojman symmetry can bring new features to cosmology and gravity theories.

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I. INTRODUCTION

The current accelerated expansion of the universe could be due to an unknown energy component (dark energy) or a modification to general relativity (modified gravity) [1, 2]. In the literature, various modified gravity theories were proposed to account for the cosmic acceleration, such as $f(R)$ theory [2–4, 45, 48], scalar-tensor theory [4, 5], braneworld model [6, 7], Galileon gravity [8, 9], Gauss-Bonnet gravity [10], $f(T)$ theory [11, 12], massive gravity [13, 14]. Nowadays, modified gravity theories have been one of the main fields in modern cosmology.

As one of the leading modified gravity theories, $f(R)$ theory was proposed by generalizing the well-known Einstein-Hilbert Lagrangian R used in general relativity (GR) to an arbitrary function $f(R)$, where R is the scalar curvature. In fact, $f(R)$ theory has been extensively studied in the literature for many years (see e.g. [2–4] for reviews). It can be used to drive inflation (see e.g. [15]), play the role of dark matter (see e.g. [16]), or drive the current accelerated expansion of the universe as an competitive alternative of dark energy (see e.g. [17, 18]).

Note that there exist two different types of $f(R)$ theories in the literature (see e.g. [2–4]), namely $f(R)$ theory in the metric formalism, and $f(R)$ theory in the Palatini formalism. In the metric formalism, the affine connection $\Gamma_{\alpha\beta}^{\lambda}$ depends on the metric $g_{\mu\nu}$, and hence the field equations are derived by the variation of the action with respect to the metric $g_{\mu\nu}$ only. On the other hand, in the Palatini formalism, the affine connection $\Gamma_{\alpha\beta}^{\lambda}$ and the metric $g_{\mu\nu}$ are treated as independent variables when one varies the action. As is well known, in the case of GR (namely $f(R) \propto R$), the field equations are completely identical in these two formalisms. However, in the case of non-linear $f(R)$, the field equations are different in these two formalisms. So, the metric and Palatini $f(R)$ theories should be considered separately.

It is of interest to find the exact cosmological solutions of $f(R)$ theories. Besides other methods (e.g. reconstruction [47]), symmetry has been proved as a powerful tool to find exact solutions. On the other hand, symmetry might hint the deep physical structure of a theory, and hence considering symmetry is also well motivated. As is well known, Noether symmetry has been extensively used in cosmology and gravity theories, for instance, scalar field cosmology [19, 20], $f(R)$ theory [21–25, 46, 48], scalar-tensor theory [26, 27], $f(T)$ theory [28, 29], Gauss-Bonnet gravity [30], non-minimally coupled cosmology [31], and others [32]. It is worth noting that a (point-like) Lagrangian should be given *a priori* when one uses Noether symmetry.

In this work, we are interested to consider the so-called Hojman symmetry in $f(R)$ theories, and find the corresponding exact cosmological solutions of $f(R)$ theories via Hojman symmetry. Unlike Noether conservation theorem, the symmetry vectors and the corresponding conserved quantities in Hojman conservation theorem can be obtained by using the equations of motion directly, without using Lagrangian or Hamiltonian. In general, its conserved quantities and the exact solutions can be quite different from the ones via Noether symmetry. In fact, recently Hojman symmetry has been used in cosmology and gravity theory [33–35]. It is found that Hojman symmetry exists for a wide range of the potential $V(\phi)$ of quintessence [33] and scalar-tensor theory [34], and the corresponding exact cosmological solutions have been obtained. While Noether symmetry exists only for exponential potential $V(\phi)$ [19, 26, 27], Hojman symmetry can exist for a wide range of potentials $V(\phi)$, including not only exponential but also power-law, hyperbolic, logarithmic and other complicated potentials [33, 34]. On the other hand, it is also found that Hojman symmetry exists in $f(T)$ theory and the corresponding exact cosmological solutions are obtained [35]. The functional form of $f(T)$ is restricted to be the power-law or hypergeometric type, while the universe experiences a power-law or hyperbolic expansion. These results are also different from the ones obtained by using Noether symmetry in $f(T)$ theory [28]. Therefore, although some exact cosmological solutions of $f(R)$ theories were found by using Noether symmetry in the literature [21–25], it is still interesting to find them by using Hojman symmetry instead, because as mentioned above one can expect that the solutions via Hojman symmetry might be quite different. On the other hand, as an important lesson from history, we consider that it is better to keep an open mind to this new symmetry, especially if it can bring something new.

The so-called Hojman symmetry was proposed in the year 1992 [36]. Following [36] and e.g. [33–35], we consider a set of second order differential equations

$$\ddot{q}^i = \mathcal{F}^i(q^j, \dot{q}^j, t), \quad i, j = 1, \dots, n, \quad (1)$$

where a dot denotes a derivative with respect to time t . If $X^i = X^i(q^j, \dot{q}^j, t)$ is a symmetry vector for

Eq. (1), it satisfies [37, 38]

$$\frac{d^2 X^i}{dt^2} - \frac{\partial \mathcal{F}^i}{\partial q^j} X^j - \frac{\partial \mathcal{F}^i}{\partial \dot{q}^j} \frac{dX^j}{dt} = 0, \quad (2)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \mathcal{F}^i \frac{\partial}{\partial \dot{q}^i}. \quad (3)$$

The symmetry vector X^i is defined so that the infinitesimal transformation

$$\hat{q}^i = q^i + \epsilon X^i(q^j, \dot{q}^j, t) \quad (4)$$

maps solutions q^i of Eq. (1) into solutions \hat{q}^i of the same equations (up to ϵ^2 terms) [37, 38]. If the “force” \mathcal{F}^i satisfies (in some coordinate systems)

$$\frac{\partial \mathcal{F}^i}{\partial \dot{q}^i} = -\frac{d}{dt} \ln \gamma, \quad (5)$$

where $\gamma = \gamma(q^i)$ is a function of q^i , then

$$Q = \frac{1}{\gamma} \frac{\partial (\gamma X^i)}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left(\frac{dX^i}{dt} \right) \quad (6)$$

is a conserved quantity for Eq. (1), namely $dQ/dt = 0$. Note that in the case of $\gamma = \text{const.}$, Eqs. (5) and (6) become simple and trivial. In the proof of Hojman conservation theorem [36] (see also e.g. [39]), neither a Lagrangian nor a Hamiltonian is needed, and no previous knowledge of a constant of motion for system (1) is invoked either [36]. In this way, Hojman conservation theorem is different from Noether conservation theorem.

In the present work, we consider Hojman symmetry in $f(R)$ theories in both the metric and Palatini formalisms, and find the corresponding exact cosmological solutions. In fact, they are the main contents of Secs. II and III, respectively. One can expect new results by using Hojman symmetry in $f(R)$ theories. The brief concluding remarks are given in Sec. IV.

II. EXACT COSMOLOGICAL SOLUTIONS OF $f(R)$ THEORY IN THE METRIC FORMALISM

In this section, we consider $f(R)$ theory in the metric formalism at first. The action is given by

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + \mathcal{S}_M, \quad (7)$$

where $\kappa^2 \equiv 8\pi G$, g is the determinant of the metric $g_{\mu\nu}$, and \mathcal{S}_M is the matter action. In the metric formalism, the affine connection $\Gamma_{\alpha\beta}^\lambda$ depends on the metric $g_{\mu\nu}$, and hence the field equations are derived by the variation of the action with respect to the metric $g_{\mu\nu}$. Throughout this work, we consider a spatially flat Friedmann-Robertson-Walker (FRW) universe whose spacetime is described by

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2, \quad (8)$$

where a is the scale factor. For this metric, in the metric formalism, the Ricci scalar R is given by [2–4]

$$R = 6 \left(2H^2 + \dot{H} \right), \quad (9)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and a dot denotes a derivative with respect to cosmic time t . The modified Friedmann equations read [2–4]

$$3FH^2 = (FR - f)/2 - 3H\dot{F} + \kappa^2 \rho_M, \quad (10)$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + \kappa^2 (\rho_M + p_M), \quad (11)$$

where $F = f_{,R} \equiv \partial f / \partial R$, and ρ_M , p_M are the energy density and pressure of matter, respectively. The energy conservation equation of matter is given by

$$\dot{\rho}_M + 3H(\rho_M + p_M) = 0. \quad (12)$$

The equation-of-state parameter (EoS) of matter is defined by $w_M = p_M / \rho_M$. In particular, $w_M = 0$ and $1/3$ correspond to pressureless matter and radiation, respectively.

The main difficulty to consider Hojman symmetry in the metric $f(R)$ theory is that the corresponding equations of motion (Eqs. (10) and (11)) are 4th order with respect to the scale factor a , while Hojman symmetry deals with 2nd order equations as mentioned in Sec. I. We should try to recast them as second order differential equations. Inspired by the well-known conformal transformation [2–4], we introduce new variables \tilde{t} and \tilde{a} according to

$$d\tilde{t} = \sqrt{F} dt, \quad \tilde{a} = \sqrt{F} a. \quad (13)$$

While the traditional conformal transformation mainly deals with the Lagrangian/action, here we instead directly deal with the equations of motion using Eq. (13). Also, we introduce

$$\tilde{H} \equiv \frac{1}{\tilde{a}} \frac{d\tilde{a}}{d\tilde{t}} = \frac{1}{\sqrt{F}} \left(H + \frac{\dot{F}}{2F} \right), \quad (14)$$

in which we have used Eq. (13). Introducing a new scalar field ϕ according to

$$\kappa\phi = \sqrt{\frac{3}{2}} \ln F, \quad (15)$$

we can recast Eq. (10) as

$$\tilde{H}^2 = \frac{\kappa^2}{3} (\tilde{\rho}_\phi + \tilde{\rho}_M), \quad (16)$$

where $\tilde{\rho}_M = \rho_M / F^2$, and

$$\tilde{\rho}_\phi = \frac{1}{2} \left(\frac{d\phi}{d\tilde{t}} \right)^2 + V(\phi), \quad V(\phi) = \frac{FR - f}{2\kappa^2 F^2}. \quad (17)$$

By the help of Eqs. (9) and (10), we can recast Eq. (11) as the equation of motion for the scalar field ϕ , namely

$$\frac{d^2\phi}{d\tilde{t}^2} + 3\tilde{H} \frac{d\phi}{d\tilde{t}} + V_{,\phi} = \frac{\kappa}{\sqrt{6}} (\tilde{\rho}_M - 3\tilde{p}_M), \quad (18)$$

where $\tilde{p}_M = p_M / F^2$, and $V_{,\phi} = \partial V / \partial \phi$. In fact, Eq. (18) is equivalent to

$$\frac{d\tilde{\rho}_\phi}{d\tilde{t}} + 3\tilde{H} (\tilde{\rho}_\phi + \tilde{p}_\phi) = \frac{\kappa}{\sqrt{6}} (\tilde{\rho}_M - 3\tilde{p}_M) \frac{d\phi}{d\tilde{t}}, \quad (19)$$

where

$$\tilde{p}_\phi = \frac{1}{2} \left(\frac{d\phi}{d\tilde{t}} \right)^2 - V(\phi). \quad (20)$$

On the other hand, Eq. (12) can be recast as

$$\frac{d\tilde{\rho}_M}{d\tilde{t}} + 3\tilde{H} (\tilde{\rho}_M + \tilde{p}_M) = -\frac{\kappa}{\sqrt{6}} (\tilde{\rho}_M - 3\tilde{p}_M) \frac{d\phi}{d\tilde{t}}. \quad (21)$$

So, the “total energy conservation equation” holds, namely

$$\frac{d\tilde{\rho}_{tot}}{d\tilde{t}} + 3\tilde{H}(\tilde{\rho}_{tot} + \tilde{p}_{tot}) = 0, \quad (22)$$

where $\tilde{\rho}_{tot} = \tilde{\rho}_\phi + \tilde{\rho}_M$, and $\tilde{p}_{tot} = \tilde{p}_\phi + \tilde{p}_M$. Using Eqs. (16) and (22), we obtain

$$\frac{1}{\tilde{a}} \frac{d^2 \tilde{a}}{d\tilde{t}^2} = -\frac{\kappa^2}{6} (\tilde{\rho}_{tot} + 3\tilde{p}_{tot}). \quad (23)$$

Now, we try to consider Hojman symmetry in the metric $f(R)$ theory. Following [33–35], we introduce a new variable $\tilde{x} \equiv \ln \tilde{a}$. From now on, in order to make the expressions simple, we use an empty circle “o” to denote a derivative with respect to the “new time” \tilde{t} . So, it is easy to see $\overset{\circ}{\tilde{x}} = \tilde{H}$. Using Eqs. (23) and (16), we have

$$\overset{\circ}{\tilde{x}} = \tilde{H} = \frac{d^2 \tilde{x}}{d\tilde{t}^2} = -\frac{\kappa^2}{2} (\tilde{\rho}_\phi + \tilde{p}_\phi + \tilde{\rho}_M + \tilde{p}_M). \quad (24)$$

Following [33, 34], here we only consider the “dark energy” dominated epoch, and hence the contributions from matter can be ignored. For convenience, we also set the unit $\kappa = 1$. Noting Eqs. (17) and (20), it is easy to see that Eq. (24) becomes

$$\overset{\circ}{\tilde{x}} = -s(\tilde{x}) \overset{\circ}{\tilde{x}}^2 = \mathcal{F}(\tilde{x}, \overset{\circ}{\tilde{x}}), \quad (25)$$

where

$$s(\tilde{x}) = \frac{1}{2} \phi'^2(\tilde{x}), \quad (26)$$

and a prime denotes a derivative with respect to the variable of the function, namely $h'(y) = dh(y)/dy$. Note that Eqs. (25) and (26) in this work have the same forms as Eqs. (21) and (22) of [33], except for the different variables \tilde{x} and \tilde{t} . Therefore, the derivations below are straightforward by using the needed results from [33]. If Hojman symmetry exists in this theory, the condition (5) should be satisfied. From Eqs. (25), (5) and (3) replaced t with \tilde{t} , we find that

$$\gamma(\tilde{x}) = \gamma_0 \exp \left(2 \int s(\tilde{x}) d\tilde{x} \right), \quad (27)$$

where γ_0 is an integration constant. Following [33–35], we assume that the symmetry vector X does not explicitly depend on time \tilde{t} . Then, Eq. (2) replaced t with \tilde{t} becomes [33]

$$\left[s(\tilde{x}) \frac{\partial X}{\partial \tilde{x}} + s'(\tilde{x}) X + \frac{\partial^2 X}{\partial \tilde{x}^2} \right] + \overset{\circ}{\tilde{x}}^2 s^2(\tilde{x}) \frac{\partial^2 X}{\partial \tilde{x}^2} - \overset{\circ}{\tilde{x}} \left[2s(\tilde{x}) \frac{\partial^2 X}{\partial \tilde{x} \partial \tilde{x}} + s'(\tilde{x}) \frac{\partial X}{\partial \tilde{x}} \right] = 0. \quad (28)$$

Using Eqs. (16), (18), and ignoring the contributions from matter, we have [33]

$$\frac{V'(\phi)}{V(\phi)} = \frac{s(\tilde{x})\phi'(\tilde{x}) - \phi''(\tilde{x}) - 3\phi'(\tilde{x})}{3 - \frac{1}{2}\phi'^2(\tilde{x})}, \quad (29)$$

which is useful to derive the potential $V(\phi)$.

A. Power-law solution

In fact, the differential equation for the symmetry vector X , namely Eq. (28), is difficult to solve in general. The authors of [33] had tried various ansatz for the symmetry vector $X(\tilde{x}, \overset{\circ}{\tilde{x}})$. For the ansatz

$$X(\tilde{x}, \overset{\circ}{\tilde{x}}) = \overset{\circ}{\tilde{x}} g(\tilde{x}), \quad \text{and} \quad g(\tilde{x}) = \lambda \exp \left(\frac{\alpha^2}{2} \tilde{x} \right), \quad (30)$$

the corresponding cosmological solutions obtained in [33] are given by

$$V(\varphi) = \frac{2(6 - \alpha^2)Q_0^2}{\alpha^4\lambda^2} e^{\mp\alpha\varphi}, \quad (31)$$

$$\tilde{a}(\tilde{t}) = e^{\tilde{x}_0} \left[1 + \frac{Q_0\alpha^2}{2\lambda} \exp\left(-\frac{\alpha^2}{2}\tilde{x}_0\right) (\tilde{t} - \tilde{t}_0) \right]^{2/\alpha^2}, \quad (32)$$

$$\varphi(\tilde{t}) = \pm \frac{2}{\alpha} \ln \left[1 + \frac{Q_0\alpha^2}{2\lambda} \exp\left(-\frac{\alpha^2}{2}\tilde{x}_0\right) (\tilde{t} - \tilde{t}_0) \right], \quad (33)$$

where $\varphi = \phi - \phi_0$, and $\phi_0, \tilde{t}_0, \tilde{x}_0, \lambda, \alpha$ are constants. The conserved quantity is given by [33]

$$\dot{\tilde{x}}g'(\tilde{x}) = Q_0 = \text{const.} \quad (34)$$

We refer to [33] for the detailed derivations. With these results, we can convert them into the cosmological solutions in the metric $f(R)$ theory. For convenience, we recast Eq. (31) as $V(\phi) = V_0 e^{\mp\alpha\phi}$, where $V_0 = 2(6 - \alpha^2)Q_0^2 \exp(\pm\phi_0)/(\alpha^4\lambda^2)$ is constant. Using this $V(\phi)$ and Eqs. (17), (15), we have

$$FR - f = 2V_0 F^\beta, \quad (35)$$

where $\beta = 2 \mp \sqrt{\frac{3}{2}}\alpha$. Noting that $F = f_{,R} \equiv \partial f / \partial R$, it is a differential equation for $f(R)$ with respect to R in fact. It is easy to find the solution as

$$f(R) = c_1 R^n, \quad (36)$$

where $n = \beta/(\beta - 1)$ and $c_1 = ((n - 1)/(2V_0 n^\beta))^{1/(\beta - 1)}$ are both constants. In the case of $n = 1$, the solution reads $f(R) = c_2 R - 2c_2^\beta V_0$ where c_2 is constant. Since it is trivial, we do not consider the case of $n = 1$ any more. Let us turn to find the scale factor $a(t)$ and the Hubble parameter $H(t)$ or $H(a)$. Using Eqs. (15) and (33), we get

$$\sqrt{F} = \exp\left(\frac{\phi_0}{\sqrt{6}}\right) [1 + c_0 (\tilde{t} - \tilde{t}_0)]^{\pm\sqrt{\frac{2}{3}}/\alpha}, \quad (37)$$

where $c_0 = (Q_0\alpha^2/(2\lambda)) \exp(-\alpha^2\tilde{x}_0/2)$ is constant. Integrating $dt = d\tilde{t}/\sqrt{F}$ from Eq. (13) gives

$$t - t_0 = c_{31} [1 + c_0 (\tilde{t} - \tilde{t}_0)]^{1 \mp \sqrt{\frac{2}{3}}/\alpha}, \quad \text{or} \quad 1 + c_0 (\tilde{t} - \tilde{t}_0) = c_{32} (t - t_0)^{1/(1 \mp \sqrt{\frac{2}{3}}/\alpha)}, \quad (38)$$

where $c_{31} = \exp(-\phi_0/\sqrt{6})/(c_0(1 \mp \sqrt{2/3}/\alpha))$, $c_{32} = c_{31}^{-1/(1 \mp \sqrt{2/3}/\alpha)}$, and t_0 is an integration constant. Substituting Eq. (38) into Eqs. (32), (37) and then $a = \tilde{a}/\sqrt{F}$ from Eq. (13), we obtain

$$a(t) = c_3 (t - t_0)^m, \quad (39)$$

where $m = (2/\alpha^2 \mp \sqrt{2/3}/\alpha)/(1 \mp \sqrt{2/3}/\alpha)$ and $c_3 = \exp(\tilde{x}_0 - \phi_0/\sqrt{6}) c_{32}^{2/\alpha^2 \mp \sqrt{2/3}/\alpha}$ are both constants. Obviously, the universe experiences a power-law expansion. Note that this solution can also be found via Noether symmetry [21–23]. From Eq. (39), it is easy to obtain the Hubble parameter as

$$H(t) = \frac{\dot{a}}{a} = m (t - t_0)^{-1}, \quad \text{or} \quad H(a) = H_0 a^{-1/m}, \quad (40)$$

where $H_0 = mc_3^{1/m}$ is the Hubble constant.

B. New solutions

In [33], other ansatz for the symmetry vector X are also considered. For the ansatz

$$X = X(\tilde{x}) = A_0 \tilde{x}^{-1/\alpha}, \quad (41)$$

the corresponding cosmological solutions obtained in [33] are given by

$$V(\varphi) = \lambda\varphi^{-4\alpha} - \frac{8}{3}\lambda\alpha^2\varphi^{-4\alpha-2}, \quad (42)$$

$$\tilde{a}(\tau) = e^{\alpha s_0} \exp\left(\alpha((1+\alpha)\tau)^{1/(1+\alpha)}\right), \quad (43)$$

$$\varphi(\tau) = \mp\sqrt{8}\alpha[(1+\alpha)\tau]^{1/(2(1+\alpha))}, \quad (44)$$

where $\varphi = \phi - \phi_c$, $\tau = y_0 + \alpha^{-1}|Q_0|^{-\alpha}\tilde{t}$, and $\phi_c, y_0, s_0, \lambda, \alpha$ are constants. The conserved quantity is given by [33]

$$\frac{\dot{\tilde{x}}^{-1/\alpha}}{s_0 - \tilde{x}/\alpha} = Q_0 = \text{const.} \quad (45)$$

Note that the same solutions (42)–(44) can also be found by using another ansatz [33]

$$X(\tilde{x}, \dot{\tilde{x}}) = \dot{\tilde{x}} g(\tilde{x}), \quad \text{and} \quad g(\tilde{x}) = \frac{(f_0 + \tilde{x})^{1+\alpha}}{1+\alpha}. \quad (46)$$

We refer to [33] for the detailed derivations. With these results, we can convert them into the cosmological solutions in the metric $f(R)$ theory. Using Eqs. (15) and (44), we get

$$\sqrt{F} = \exp\left(\frac{\phi_c}{\sqrt{6}}\right) \exp(c_0 \tilde{\tau}^\beta), \quad (47)$$

where $\beta = 1/(2(1+\alpha))$, $c_0 = \mp 2\alpha/\sqrt{3}$ are both constants, and $\tilde{\tau} = (1+\alpha)\tau = \tilde{y}_0 + \tilde{Q}_0\tilde{t}$. Substituting Eq. (47) into $d\tilde{t} = \sqrt{F}dt$ from Eq. (13), we have

$$\frac{d\tilde{\tau}}{dt} = c_{21} \exp(c_0 \tilde{\tau}^\beta), \quad (48)$$

where $c_{21} = \tilde{Q}_0 \exp(\phi_c/\sqrt{6})$ is constant. The solution of Eq. (48) is given by

$$t - t_0 = -\frac{\tilde{\tau}}{\beta c_{21}} E_{\frac{\beta-1}{\beta}}(c_0 \tilde{\tau}^\beta), \quad (49)$$

where t_0 is an integration constant, and $E_n(z) = \int_1^\infty e^{-zu} u^{-n} du$ is the exponential integral function. Unfortunately, if $\beta \neq 1$, it is hard to find $\tilde{\tau}$ as a function of t by solving Eq. (49), and hence it is also hard to find the scale factor a as a function of t . Therefore, we only consider the case of $\beta = 1$ here (note that $\beta = 1$ corresponds to $\alpha = -1/2$ actually). In this case, the solution of Eq. (48) reads

$$\tilde{\tau} = -\frac{1}{c_0} \ln(t - t_0) + c_2, \quad (50)$$

where $c_2 = -c_0^{-1} \ln(-c_0 c_{21})$ is constant. Substituting Eq. (50) into Eqs. (43), (47) and then $a = \tilde{a}/\sqrt{F}$ from Eq. (13), noting that $\beta = 1$ (namely $\alpha = -1/2$), we have

$$a(t) = c_3 (t - t_0)^{1+c_2/c_0} \exp\left(-\frac{1}{2c_0^2} (\ln(t - t_0))^2\right), \quad (51)$$

where $c_3 = \exp(-s_0/2 - \phi_c/\sqrt{6} - c_0 c_2 - c_2^2/2)$ is constant. *Obviously, it is not a power-law solution. To our knowledge, this new solution has not been found previously in the literature.* From Eq. (51), it is easy to obtain the Hubble parameter as

$$H(t) = \frac{\dot{a}}{a} = c_0^{-2} (t - t_0)^{-1} (c_4 - \ln(t - t_0)), \quad (52)$$

where $c_4 = c_0(c_0 + c_2)$ is constant. From Eq. (51), we find

$$\ln(t - t_0) = \eta(a) = c_4 \pm \sqrt{c_4^2 - 2 \ln \frac{a}{c_3}}. \quad (53)$$

Substituting it into Eq. (52), we get

$$H(a) = c_0^{-2} (c_4 - \eta(a)) e^{-\eta(a)}. \quad (54)$$

Note that in the case of $\beta = 1$ (namely $\alpha = -1/2$), the corresponding $c_0^{-2} = 3$ in fact. Next, let us turn to find $f(R)$ as a function of R . Using Eqs. (17), (15), and (42) with $\alpha = -1/2$, we obtain

$$f = FR - 2\lambda F^2 \left[\left(\sqrt{\frac{3}{2}} \ln F - \phi_c \right)^2 - \frac{2}{3} \right]. \quad (55)$$

Noting that $F = f_{,R} \equiv \partial f / \partial R$, it is a differential equation for $f(R)$ with respect to R . Unfortunately, this differential equation is hard to solve in general. Since the solution $f(R) = c_{10}R + c_{20}$ is trivial, we should find a way to obtain the non-trivial solution. Substituting Eq. (52) into Eq. (9), we get

$$R(t) = -6c_0^{-4} (t - t_0)^{-2} [-2c_4^2 + c_0^2(1 + c_4) - \ln(t - t_0)(c_0^2 - 4c_4 + 2 \ln(t - t_0))] . \quad (56)$$

Substituting Eq. (50) into Eq. (47) with $\beta = 1$, we obtain

$$\sqrt{F} = \exp\left(\frac{\phi_c}{\sqrt{6}}\right) \exp(c_0 c_2 - \ln(t - t_0)) . \quad (57)$$

Substituting Eqs. (56) and (57) into Eq. (55), $f(t)$ can be found as a function of time t . Unfortunately, it is hard to solve Eq. (56) and obtain $t - t_0$ or $\ln(t - t_0)$ as an explicit function of R . So, we cannot obtain $f(R)$ as an explicit function of R . Nevertheless, with $f(t)$ and $R(t)$, we can still regard $f(t) = f(t(R)) = f(R)$ as an implicit function in principle.

Note that other exotic ansatz for the symmetry vector X are considered in [33], and the corresponding $V(\phi)$, $\tilde{a}(\tilde{t})$ and $\phi(\tilde{t})$ are found. However, it is hard to obtain $f(R)$, $a(t)$ and $H(t)$ in these cases. Although further complicated ansatz for the symmetry vector X beyond the ones in [33] could be tried, we stop here. Let us turn to $f(\mathcal{R})$ theory in the Palatini formalism.

III. EXACT COSMOLOGICAL SOLUTIONS OF $f(\mathcal{R})$ THEORY IN THE PALATINI FORMALISM

In this section, we consider $f(\mathcal{R})$ theory in the Palatini formalism. The action is given by

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(\mathcal{R}) + \mathcal{S}_M, \quad (58)$$

where $\kappa^2 \equiv 8\pi G$, g is the determinant of the metric $g_{\mu\nu}$, and \mathcal{S}_M is the matter action. In the Palatini formalism, the affine connection $\Gamma_{\alpha\beta}^\lambda$ and the metric $g_{\mu\nu}$ are treated as independent variables. So, the Ricci scalar \mathcal{R} is different from the one in the metric formalism, and their relation reads [2–4, 24, 25, 40–42]

$$\mathcal{R} = R + \frac{3}{2F^2} \nabla_\mu F \nabla^\mu F - \frac{3}{F} \square F, \quad (59)$$

where $F = f_{,\mathcal{R}} \equiv \partial f / \partial \mathcal{R}$. Thus, in the Palatini formalism $\mathcal{R} \neq R = 6(2H^2 + \dot{H})$ generally. We consider a spatially flat FRW universe whose spacetime is described by Eq. (8), and the modified Friedmann equations read [2–4, 24, 25, 40–42]

$$F\mathcal{R} - 2f = -\kappa^2 (\rho_M - 3p_M), \quad (60)$$

$$6F \left(H + \frac{\dot{F}}{2F} \right)^2 - f = \kappa^2 (\rho_M + 3p_M). \quad (61)$$

The energy conservation equation of matter is given by Eq. (12). In this section, we do not ignore the contributions from matter. To be simple, here we only consider the case of $w_M = p_M/\rho_M = 0$, namely pressureless matter. Differentiating Eq. (60) and using Eq. (12) with $p_M = 0$, one has [2–4, 40]

$$\dot{\mathcal{R}} = \frac{3\kappa^2 H \rho_M}{F_{,\mathcal{R}} \mathcal{R} - F} = -3H \cdot \frac{F\mathcal{R} - 2f}{F_{,\mathcal{R}} \mathcal{R} - F}. \quad (62)$$

Similar to Sec. II, we introduce new variables \tilde{t} and \tilde{a} according to Eq. (13), and obtain \tilde{H} in Eq. (14) by definition. Adding Eqs. (60) and (61) with $p_M = 0$, we have

$$\tilde{H}^2 = \frac{3f - F\mathcal{R}}{6F^2}. \quad (63)$$

Following [33–35], we introduce a new variable $\tilde{x} \equiv \ln \tilde{a}$, and use an empty circle “ \circ ” to denote a derivative with respect to the “new time” \tilde{t} . So, it is easy to see $\dot{\tilde{x}} = \tilde{H}$. Since the right hand side of Eq. (63) is a function explicitly depends only on \mathcal{R} . Thus, $\mathcal{R} = \mathcal{R}(\tilde{x})$ is a function explicitly depends only on $\tilde{x} = \tilde{H}$. Differentiating Eq. (63) with respect to \tilde{t} , and using Eqs. (62), (13), (14), we obtain

$$\dot{\tilde{H}} = \frac{F\mathcal{R} - 2f}{2F^2}. \quad (64)$$

Noting that $\mathcal{R} = \mathcal{R}(\tilde{x})$, its right hand side is a function explicitly depends only on \tilde{x} . Therefore, Eq. (64) is in fact

$$\dot{\tilde{x}} = \dot{\tilde{H}} = \frac{F\mathcal{R} - 2f}{2F^2} = \mathcal{F}(\tilde{x}). \quad (65)$$

The “force” \mathcal{F} explicitly depends only on $\tilde{x} = \tilde{H}$. If Hojman symmetry exists in the Palatini $f(\mathcal{R})$ theory, the condition (5) should be satisfied. Noting Eq. (3) and $\gamma = \gamma(\tilde{x})$, we recast Eq. (5) as

$$-\frac{1}{\tilde{x}} \frac{\partial \mathcal{F}(\tilde{x})}{\partial \tilde{x}} = \frac{\partial}{\partial \tilde{x}} \ln \gamma(\tilde{x}). \quad (66)$$

Since its left hand side is a function of \tilde{x} only, and its right hand side is a function of \tilde{x} only, they must be equal to a same constant in order to ensure that Eq. (66) always holds. For convenience, we let this constant be $2n$, and then Eq. (66) can be separated into two ordinary differential equations

$$\frac{\partial}{\partial \tilde{x}} \ln \gamma(\tilde{x}) = 2n, \quad \frac{\partial \mathcal{F}(\tilde{x})}{\partial \tilde{x}} = -2n\tilde{x}. \quad (67)$$

Thus, it is easy to find that

$$\gamma(\tilde{x}) = \gamma_0 e^{2n\tilde{x}}, \quad (68)$$

$$\mathcal{F}(\tilde{x}) = -n\tilde{x}^2 + c_0, \quad (69)$$

where γ_0 and c_0 are both integration constants. In the following subsections, we consider the cases of $c_0 = 0$ and $c_0 \neq 0$, respectively.

A. Power-law solution with $c_0 = 0$

In the case of $c_0 = 0$, substituting Eq. (69) into Eq. (65), and using Eq. (63), we obtain

$$3(n-2)f = (n-3)F\mathcal{R}. \quad (70)$$

Noting $F = f_{,\mathcal{R}} \equiv \partial f / \partial \mathcal{R}$, it is a differential equation for $f(\mathcal{R})$ with respect to \mathcal{R} in fact. Note that if $n = 0$, from Eq. (70) we have $F\mathcal{R} - 2f = 0$, while $F\mathcal{R} - 2f = -\kappa^2 \rho_M$ from Eq. (60). So, $n \neq 0$ is required

unless $\rho_M = 0$. In the case of $n = 2$, $f(\mathcal{R}) = \text{const.}$, and in the case of $n = 3$, $f(\mathcal{R}) = 0$. Thus, we do not consider these trivial cases of $n = 2$ and 3 . In other cases, the solution of Eq. (70) is given by

$$f(\mathcal{R}) = c_1 \mathcal{R}^m, \quad (71)$$

where $m = 3(n-2)/(n-3)$ and c_1 are both constants. Substituting Eq. (69) into Eq. (65) with $c_0 = 0$, we have $\ddot{H} = -n\tilde{H}^2$ whose solution reads

$$\tilde{H}(\tilde{t}) = \frac{1}{n} (\tilde{t} + c_2)^{-1}, \quad (72)$$

where c_2 is an integration constant. From $\tilde{H} = \dot{a}/\tilde{a}$, it is easy to get

$$\tilde{a}(\tilde{t}) = c_3 (\tilde{t} + c_2)^{1/n}, \quad (73)$$

where c_3 is an integration constant. Substituting Eq. (71) into Eq. (63), and using Eq. (72), we have

$$\mathcal{R} = \left[\frac{6c_1 m^2}{n^2(3-m)} \right]^{1/(2-m)} (\tilde{t} + c_2)^{-2/(2-m)}, \quad (74)$$

and then

$$F = c_1 m \mathcal{R}^{m-1} = c_{41} (\tilde{t} + c_2)^{-2(m-1)/(2-m)}, \quad (75)$$

where $c_{41} = c_1 m [6c_1 m^2 / (n^2(3-m))]^{(m-1)/(2-m)}$ is constant. Since $n \neq 0$ and $n \neq 2$ as mentioned above, we note that $m \neq 0$, $m \neq 2$ and $m \neq 3$. Substituting Eq. (75) into $dt = d\tilde{t}/\sqrt{F}$ from Eq. (13), it is easy to obtain

$$t - t_0 = \frac{nc_{41}^{-1/2}}{3-n} (\tilde{t} + c_2)^{(3-n)/n}, \quad \text{or} \quad \tilde{t} + c_2 = c_{42} (t - t_0)^{n/(3-n)}, \quad (76)$$

where $c_{42} = [c_{41}^{1/2}(3/n-1)]^{n/(3-n)}$ is constant. Substituting Eq. (75) into $a = \tilde{a}/\sqrt{F}$ from Eq. (13), and using Eqs. (73), (76), we find that

$$a(t) = c_4 (t - t_0)^{2m/3}, \quad (77)$$

where $c_4 = c_3 c_{41}^{-1/2} c_{42}^{4/n-2}$ is constant. Obviously, the universe experiences a power-law expansion. Note that this solution can also be found via Noether symmetry [24, 25]. From Eq. (77), it is easy to get the Hubble parameter as

$$H(t) = \frac{\dot{a}}{a} = \frac{2m}{3} (t - t_0)^{-1}, \quad \text{or} \quad H(a) = H_0 a^{-3/(2m)}, \quad (78)$$

where $H_0 = (2m/3) c_4^{3/(2m)}$ is the Hubble constant.

Let us turn to the conserved quantity. Following [33–35], we assume that the symmetry vector X does not explicitly depend on time. Substituting Eq. (69) with $c_0 = 0$ into Eq. (2), the equation for X reads

$$\frac{\partial^2 X}{\partial \tilde{x}^2} - 2n\tilde{x} \frac{\partial^2 X}{\partial \tilde{x} \partial \tilde{x}} + n^2 \tilde{x}^2 \frac{\partial^2 X}{\partial \tilde{x}^2} + n \frac{\partial X}{\partial \tilde{x}} = 0. \quad (79)$$

To solve this equation, we adopt the ansatz

$$X = A_0 \tilde{x}^\alpha e^{\beta \tilde{x}} + A_1, \quad (80)$$

where A_0, A_1, α, β are all constants, and α, β cannot be zero at the same time. Substituting Eq. (80) into Eq. (79), we find that the solutions have $n\alpha - \beta = 0$ or $n\alpha - \beta = n$. Substituting Eqs. (80) and (68) into Eq. (6), the conserved quantity Q is given by

$$Q = 2nA_1 - (2 + \alpha)(n\alpha - \beta - n)A_0 \tilde{x}^\alpha e^{\beta \tilde{x}}. \quad (81)$$

If $n\alpha - \beta = n$ or $\alpha = -2$, then $Q = 2nA_1 = \text{const.}$ is trivial. If $n\alpha - \beta = 0$ and $\alpha \neq -2$, we get

$$\tilde{x} e^{n\tilde{x}} = \text{const.} \quad (82)$$

In fact, this conserved quantity can be found in another way. Noting that $\dot{\tilde{x}} = \tilde{H}$, $\tilde{x} = \ln \tilde{a}$, and using Eqs. (72), (73), one can find the same conserved quantity given in Eq. (82) again. This can be regarded as a confirmation of Hojman conservation theorem.

B. New solutions with $c_0 \neq 0$

In the case of $c_0 \neq 0$, substituting Eq. (69) into Eq. (65), and using Eq. (63), we obtain

$$3(n-2)f = 6c_0F^2 + (n-3)F\mathcal{R}. \quad (83)$$

Noting $F = f_{,\mathcal{R}} \equiv \partial f / \partial \mathcal{R}$, it is a differential equation for $f(\mathcal{R})$ with respect to \mathcal{R} in fact. If $n = 2$, its solutions are $f(\mathcal{R}) = \text{const.}$ or $f(\mathcal{R}) = \mathcal{R}^2/(12c_0) + \text{const.}$, which lead to $F\mathcal{R} - 2f = \text{const.}$, while $F\mathcal{R} - 2f = -\kappa^2\rho_M$ from Eq. (60). So, $n \neq 2$ is required unless $\rho_M = \text{const.}$ If $n = 3$, the solution reads $f(\mathcal{R}) = \mathcal{R}^2/(8c_0) \pm c_1\mathcal{R}/\sqrt{8c_0+c_1^2}/4$, which is trivial. If $n = 3/2$, the solution is given by $f(\mathcal{R}) = c_1\mathcal{R} + c_2$, which reduces to GR in fact. Besides these dismissed cases, we consider the cases of $n \neq 0$ and $n = 0$ one by one in the followings. In fact, some new solutions can be found via Hojman symmetry.

1. The case of $n \neq 0$

In the case of $n \neq 0$ (and also $n \neq 2, 3, 3/2$ as mentioned above), it is hard to solve Eq. (83) and obtain $f(\mathcal{R})$ as an explicit function of \mathcal{R} . In fact, from Eq. (83), $f(\mathcal{R})$ and \mathcal{R} satisfy the equation

$$\begin{aligned} 2\left(2 - \frac{3}{n}\right) \operatorname{arctanh} \left[\frac{(2n-3)\mathcal{R}}{\xi(f(\mathcal{R}), \mathcal{R})} \right] + \left(2 - \frac{3}{n}\right) \ln [24c_0f(\mathcal{R}) - n\mathcal{R}^2] \\ - 2\left(1 - \frac{3}{n}\right) \ln [2((n-3)\mathcal{R} + \xi(f(\mathcal{R}), \mathcal{R}))] = \text{const.}, \end{aligned} \quad (84)$$

or another equation

$$2\left(\frac{3}{n} - 1\right) \ln f(\mathcal{R}) - \text{left hand side of Eq. (84)} = \text{const.}, \quad (85)$$

where $\xi(f(\mathcal{R}), \mathcal{R}) = [(n-3)^2\mathcal{R}^2 + 72c_0(n-2)f(\mathcal{R})]^{1/2}$. Using Eqs. (84) or (85), we can regard $f(\mathcal{R})$ as an implicit function of \mathcal{R} in principle. Let us move forward. Substituting Eq. (69) into Eq. (65), we have $\ddot{H} = -n\dot{H}^2 + c_0$, whose solution for $c_0 \neq 0$ and $n \neq 0$ is given by

$$\tilde{H}(\tilde{t}) = \sqrt{\frac{c_0}{n}} \tanh(\sqrt{nc_0} \tilde{t} + c_2), \quad (86)$$

where c_2 is an integration constant. From $\tilde{H} = \dot{a}/\tilde{a}$, it is easy to get

$$\tilde{a}(\tilde{t}) = c_3 [\cosh(\sqrt{nc_0} \tilde{t} + c_2)]^{1/|n|}, \quad (87)$$

where c_3 is an integration constant. On the other hand, it is hard to solve Eq. (2) with $c_0 \neq 0$ to get the symmetry vector X . Thus, the task to obtain the conserved quantity Q in Eq. (6) is also hard. Fortunately, there exists another way. Inspired by the discussion below Eq. (82), using Eqs. (86) and (87), we find the conserved quantity as

$$\left(\dot{\tilde{x}}^2 - \frac{c_0}{n}\right) e^{2|n|\tilde{x}} = \text{const.}, \quad (88)$$

which can reduce to Eq. (82) if $c_0 = 0$. However, since we have no $f(\mathcal{R})$ as an explicit function of \mathcal{R} in this case, it is difficult to convert $\tilde{H}(\tilde{t})$ and $\tilde{a}(\tilde{t})$ into the cosmological solutions $H(t)$ and $a(t)$. Nevertheless, it is easy to see that $\tilde{H}(\tilde{t})$ and $\tilde{a}(\tilde{t})$ in Eqs. (86) and (87) are significantly different from the ones of power-law solution in Eqs. (72) and (73). So, it is reasonable to speculate that $H(t)$ and $a(t)$ are also not power-law. In fact, we speculate that they might be hyperbolic, akin to $\tilde{H}(\tilde{t})$ and $\tilde{a}(\tilde{t})$ in Eqs. (86) and (87). They are new solutions via Hojman symmetry. Anyway, let us turn to the case of $n = 0$.

2. The case of $n = 0$

In the case of $n = 0$, Eq. (83) becomes

$$2c_0 F^2 = F\mathcal{R} - 2f, \quad (89)$$

which is still not easy to be solved directly by itself. However, we can try to indirectly solve it by the help of Eq. (60), whose left hand side is just the right hand side of Eq. (89). Note that $\rho_M = \rho_{M0} a^{-3}$ from Eq. (12) with $p_M = 0$. Using Eqs. (89) and (60), we have $2c_0 F^2 = -\kappa^2 \rho_{M0} a^{-3}$. So, $c_0 < 0$ is required. And then, we obtain

$$F = c_4 a^{-3/2}, \quad (90)$$

where $c_4^2 = -\kappa^2 \rho_{M0} / (2c_0) > 0$ is constant. Substituting Eq. (90) into Eq. (89), we get

$$2c_0 c_4^2 a^{-3} = c_4 a^{-3/2} \mathcal{R} - 2f. \quad (91)$$

Differentiating Eq. (91) with respect to a , and noting that $f_{,a} = f_{,\mathcal{R}} \mathcal{R}_{,a} = F \mathcal{R}_{,a}$, it is easy to obtain

$$2a^{5/2} \mathcal{R}_{,a} + 2a^{3/2} \mathcal{R} = 12c_0 c_4, \quad (92)$$

which is a differential equation for \mathcal{R} with respect to a . Its solution reads

$$\mathcal{R}(a) = (6c_0 c_4 \ln a + c_{10}) a^{-3/2}, \quad (93)$$

where c_{10} is an integration constant. Substituting Eq. (93) into Eq. (91), we have

$$f(a) = \frac{1}{2} a^{-3} (6c_0 c_4^2 \ln a + c_4 c_{10} - 2c_0 c_4^2). \quad (94)$$

Solving Eq. (93), we find

$$a^{-3/2} = -\frac{\mathcal{R}}{4c_0 c_4 W(c_1 \mathcal{R})}, \quad (95)$$

where $c_1 = -\exp(-c_{10}/(4c_0 c_4))/(4c_0 c_4)$ is constant, and $W(z)$ is the Lambert W function (or product logarithm) [43], which gives the principal solution for w in $z = we^w$. Substituting Eq. (95) into Eq. (91), we obtain $f(\mathcal{R})$ as an explicit function of \mathcal{R} , namely

$$f(\mathcal{R}) = -\frac{\mathcal{R}^2}{16c_0} \left[\frac{1}{W^2(c_1 \mathcal{R})} + \frac{2}{W(c_1 \mathcal{R})} \right]. \quad (96)$$

In fact, one can check that this $f(\mathcal{R})$ indeed satisfies Eq. (89). *To our knowledge, this $f(\mathcal{R})$ has not been considered previously in the literature.*

In the case of $c_0 \neq 0$ and $n = 0$, substituting Eq. (69) into Eq. (65), we find that $\mathring{H} = c_0$, whose solution is given by

$$\tilde{H}(\tilde{t}) = c_0 \tilde{t} + c_2, \quad (97)$$

and then

$$\tilde{a}(\tilde{t}) = c_3 \exp\left(\frac{c_0}{2} \tilde{t}^2 + c_2 \tilde{t}\right), \quad (98)$$

where c_2 and c_3 are both integration constants. Substituting Eq. (90) into $\tilde{a} = \sqrt{F} a$ from Eq. (13), and using Eq. (98), it is easy to obtain

$$a = c_3^4 c_4^{-2} \exp(2c_0 \tilde{t}^2 + 4c_2 \tilde{t}). \quad (99)$$

Substituting Eqs. (90) and (99) into $d\tilde{t} = \sqrt{F} dt$ from Eq. (13), we get

$$\frac{d\tilde{t}}{dt} = c_{51} \exp\left(\frac{3}{2}c_{00}\tilde{t}^2 - 3c_2\tilde{t}\right), \quad (100)$$

where $c_{51} = \sqrt{c_4} c_3^{-3} c_4^{3/2}$ and $c_{00} = |c_0| = -c_0 > 0$ are both constants. From Eq. (100), we find that

$$\tilde{t} = \frac{c_2}{c_{00}} + \sqrt{\frac{2}{3c_{00}}} \Psi(c_5(t - t_0)), \quad (101)$$

where t_0 is an integration constant, $c_5 = c_{51}(6c_{00}/\pi)^{1/2} \exp(-3c_2^2/(2c_{00}))$ is also constant, and $\Psi(z)$ is the inverse error function $\text{erf}^{-1}(z)$ [44]. Substituting Eq. (101) into Eq. (99), we have

$$a(t) = c_6 \exp\left(-\frac{4}{3}\Psi^2(c_5(t - t_0))\right), \quad (102)$$

where $c_6 = c_3^4 c_4^{-2} \exp(2c_2^2/c_{00})$ is constant. *Obviously, it is not a power-law solution. To our knowledge, this new solution has not been found previously in the literature.* From Eq. (102), it is easy to get the Hubble parameter as

$$H(t) = \frac{\dot{a}}{a} = -\frac{4}{3}c_5\sqrt{\pi}\Psi(c_5(t - t_0)) \exp(\Psi^2(c_5(t - t_0))). \quad (103)$$

From Eqs. (102) and (103), we find that

$$H(a) = \pm \frac{2c_5\sqrt{\pi}a}{3c_6} \sqrt{-3\ln \frac{a}{c_6}}. \quad (104)$$

On the other hand, it is hard to solve Eq. (2) with $c_0 \neq 0$ to get the symmetry vector X . Thus, the task to obtain the conserved quantity Q in Eq. (6) is also hard. Fortunately, there exists another way. Inspired by the discussion below Eq. (82), using Eqs. (97) and (98), we get the conserved quantity as

$$\dot{\tilde{x}}^2 - 2c_0\tilde{x} = \text{const}. \quad (105)$$

IV. CONCLUDING REMARKS

Nowadays, $f(R)$ theory has been one of the leading modified gravity theories to explain the current accelerated expansion of the universe, without invoking dark energy. It is of interest to find the exact cosmological solutions of $f(R)$ theories. Besides other methods, symmetry has been proved as a powerful tool to find exact solutions. On the other hand, symmetry might hint the deep physical structure of a theory, and hence considering symmetry is also well motivated. As is well known, Noether symmetry has been extensively used in physics. Recently, the so-called Hojman symmetry was also considered in the literature. Unlike Noether conservation theorem, the symmetry vectors and the corresponding conserved quantities in Hojman conservation theorem can be obtained by using the equations of motion directly, without using Lagrangian or Hamiltonian. In general, its conserved quantities and the exact solutions can be quite different from the ones using Noether symmetry.

In this work, we consider Hojman symmetry in $f(R)$ theories in both the metric and Palatini formalisms, and find the corresponding exact cosmological solutions of $f(R)$ theories via Hojman symmetry. The main difficulty to consider Hojman symmetry in the metric $f(R)$ theory is that the corresponding equations of motion are 4th order with respect to the scale factor a , while Hojman symmetry deals with 2nd order equations. We should try to recast them as 2nd order differential equations. Inspired by the well-known conformal transformation, we introduce new variables \tilde{t} and \tilde{a} . This is the key idea to use Hojman symmetry in $f(R)$ theories. While the traditional conformal transformation mainly deals with the Lagrangian/action, here we instead directly deal with the equations of motion using this variable

transformation. In the Palatini $f(R)$ theory, this transformation is also employed, although the corresponding equations of motion are already 2nd order with respect to the scale factor a . We find that the equations of motion can be significantly simplified by using this variable transformation.

In both the metric and Palatini $f(R)$ theories, we can obtain the power-law cosmological solutions via Hojman symmetry, as shown in Secs. II A and III A. Note that such kind of power-law solutions can also be found by using Noether symmetry in both the metric and Palatini $f(R)$ theories (see e.g. [21–25]). However, some new solutions significantly different from the power-law solution are also obtained by using Hojman symmetry, as shown in Secs. II B and III B. *In fact, these new results cannot be found via Noether symmetry. To our knowledge, they also have not been found previously in the literature.*

Note that in the metric $f(R)$ theory, to be simple, we have only considered the “dark energy” dominated epoch following [33, 34], and hence the contributions from matter can be ignored. If we do not ignore the contributions from matter, from Eq. (24), it is easy to see that Eq. (25) should be changed to $\ddot{\tilde{x}} = -s(\tilde{x})\dot{\tilde{x}}^2 + \sigma(\tilde{x}, \dot{\tilde{x}}) = \mathcal{F}(\tilde{x}, \dot{\tilde{x}})$, where the additional term $\sigma(\tilde{x}, \dot{\tilde{x}})$ comes from $\tilde{\rho}_M = \rho_M/F^2$ and $\tilde{p}_M = p_M/F^2$. The condition (5), the equation for the symmetry vector X (namely Eq. (2)) and the equation used to derive $V(\phi)$ (namely Eq. (29)) should become more complicated. It might be a tough work to solve these equations, and we leave it as an open question.

On the other hand, in the Palatini $f(R)$ theory, we do not ignore the contributions from matter. To be simple, in Sec. III we have only considered the case of $w_M = p_M/\rho_M = 0$, namely pressureless matter. In fact, one can consider a more general case of $w_M = \text{const.}$ further. In Eqs. (60) and (61), one can incorporate $p_M = w_M \rho_M$ into the term ρ_M , namely the right hand sides of Eqs. (60) and (61) become $(1 \pm 3w_M)\rho_M$. Similarly, a factor $1 + w_M$ appears in Eq. (12). And then, such kind of constant factors containing w_M can be absorbed by redefining the other constants in the model, as done in [35]. So, it is reasonable to anticipate that the main results will not be affected, except some constants in the model might be rescaled, as shown in [35]. The detailed calculations are straightforward and trivial, and hence we do not present them here.

Finally, we would like to briefly compare the cosmological results via Noether and Hojman symmetries in various scenarios. It is worth noting that the exact solutions of $f(R)$ theory via Noether symmetry have been discussed in the comprehensive review [48] (we thank the referee for pointing out this issue). However, to our knowledge, the exact solutions of $f(R)$ theory in [48] were mainly obtained in the spherically symmetric spacetime described by

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + M(r)d\Omega_2^2, \quad (106)$$

see e.g. Sec. 15.2 of [48] (and [49]) for details. Note that $M(r) = r^2$ and $A(r) = B^{-1}(r) = 1 - 2\mathcal{M}/r$ corresponds to the Schwarzschild case of GR. The general discussions on Noether symmetry in [48] is very inspiring. However, the exact cosmological solutions in the FRW spacetime described by Eq. (8) have not been explicitly discussed in [48]. Nevertheless, the works in [48, 49] inspire us to further consider the exact solutions in the spacetime described by Eq. (106) via Hojman symmetry instead, and we leave it to the future works. On the other hand, the same authors of [48] have indeed considered the exact cosmological solutions of the metric $f(R)$ theory via Noether symmetry in the FRW spacetime described by Eq. (8) in a series of works (see e.g. [22, 50]). They found that both the exact cosmological solutions and the functional form of $f(R)$ are power-law (see also [21]). In e.g. [24, 25], the exact cosmological solutions of the Palatini $f(R)$ theory via Noether symmetry in the FRW spacetime have also been found. Again, these solutions and the functional form of $f(R)$ are also power-law. However, in the present work, we find that the exact cosmological solutions and the functional form of $f(R)$ can be *not* power-law, by using Hojman symmetry in both the metric and Palatini formalisms, as shown in Secs. II B and III B. In addition, as mentioned in Sec. I, it is found that Hojman symmetry exists for a wide range of the potential $V(\phi)$ of quintessence [33] and scalar-tensor theory [34], and the corresponding exact cosmological solutions have been obtained. While Noether symmetry exists only for exponential potential $V(\phi)$ [19, 26, 27], Hojman symmetry can exist for a wide range of potentials $V(\phi)$, including not only exponential but also power-law, hyperbolic, logarithmic and other complicated potentials [33, 34]. On the other hand, it is also found that Hojman symmetry exists in $f(T)$ theory and the corresponding exact cosmological solutions are obtained [35]. The functional form of $f(T)$ is restricted to be the power-law or hypergeometric type, while the universe experiences a power-law or hyperbolic expansion. These results are also different from the ones obtained by using Noether symmetry in $f(T)$ theory [28]. So, in summary, Hojman symmetry can bring new features to cosmology and gravity theories.

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